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Approximate Poincaré invariance of the post-Newtonian metric tensor and gauge conditions

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Abstract. Imposing an approximate Poincaré invariance on the metric tensor in the post-Newtonian scheme of general relativity, we find a recursive method which gives restrictions to the gauge conditions at each order of approximation. Those restrictions are explicitly calculated up to the second post-Newtonian approximation (PNA) for a bounded perfect fluid. At the first PNA we choose the harmonic gauge, which satisfies the imposed restrictions, but at the second PNA the harmonic gauge does not give a Poincaré-invariant metric tensor and we choose a modified harmonic gauge instead. Using these gauges, the gravitational interaction can be described by approximate Poincaré-invariant equations of motion in the framework of predictive relativistic mechanics.

1. Introduction

The equations of motion for a gravitating system can be obtained from the general relativity theory by using different approximation methods. One of them is the so-called 'slow motion' approximation.

The slow motion approximation was introduced by Einstein *et al* (1938) (referred to as EIH) who derived the equations of motion approximated up to terms of order c^{-2}, the first post-Newtonian approximation (first PNA), for a gravitationally interacting system of point masses, which are considered as singularities of the metric tensor.

A different approach was considered for the study of a bounded fluid by Chandrasekhar (1965), who derived, in the first PNA, the hydrodynamic equations for a relativistic perfect fluid. Using Chandrasekhar's post-Newtonian method, the equations of motion for point masses of EIH can also be rederived (Spyrou 1975).

As is well known, the equations of motion in the first PNA are Poincaré invariant in the sense that under a post-Galilean, Poincaré-like coordinate transformation the equations of motion in the new coordinates have the same functional form as in the old coordinates. For a discussion of the post-Galilean transformations for a system of point masses, see Chandrasekhar and Contopoulos (1967), and for a perfect fluid system see Spyrou (1976).

Such invariance makes possible the description of the gravitational interaction, in an approximate way (up to order c^{-2}), within a special relativity theory like predictive relativistic mechanics (PRM). This theory was developed by Currie (1966), Hill (1967) and Bel (1970), and in it the interactions between particles are described by second-order differential equations of motion.

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In a previous paper (Verdaguer 1978, hereafter referred to as I) it was shown that the gravitational interaction for a system of point masses can also be described in the framework of PRM up to order c^{-4} (the second PNA). That was achieved by solving the Einstein field equations in the de Donder coordinate condition (harmonic gauge), in which the equations of motion of the system are second-order differential equations and Poincaré invariant in the sense described above. It must be pointed out that in a PNA scheme at a given order, we always get second-order differential equations of motion because higher-order time derivatives of the positions can be removed using the equations of motion at lower orders. On the other hand, the Poincaré invariance of those equations is a direct consequence of the particular gauge imposed. The imposition of a gauge is a requirement of the covariance of the Einstein field equations.

In this paper we want to discuss the direct relation between the Poincaré invariance of the metric tensor and the gauge conditions in a post-Newtonian scheme of general relativity. Invariance of the metric tensor at a given order under a coordinate transformation implies invariance of the equations of motion at the same order under the same coordinate transformation, but in a lower order in the time-coordinate part of the transformation. This is due to the non-symmetric role played by the temporal and spatial coordinates in the slow-motion approximation scheme. For instance, as was shown in I, the metric tensor up to second PNA using the harmonic gauge was not Poincaré invariant even if the equations of motion were. The same is true in the first PNA metric tensor, which is not Poincaré invariant in the gauge used by Chandrasekhar and Contopoulos, while the equations of motion are.

We follow the post-Newtonian scheme of Chandrasekhar for a bounded perfect fluid and we devote our attention to the near zone only, where the source fluid is located.

In § 2, starting with a post-Newtonian scheme for Einstein's field equations, we define the gauge functions which must be used to impose coordinate conditions at each order of the approximation. Imposing Poincaré invariance up to order N, we find recursive relations for the gauge functions and conclude that, in order to obtain a Poincaré-invariant metric tensor at each level of approximation, only gauge functions satisfying the mentioned relations can be imposed on the field equations.

In § 3, we apply those relations for a bounded perfect fluid up to the first PNA. It is found that there is an infinite set of gauge functions in which the first PNA metric tensor is Poincaré invariant, the harmonic gauge function at this order belonging to this set. Obviously the gauge used by Chandrasekhar and Contopoulos does not satisfy the required relations.

In § 4, having chosen the harmonic coordinate condition in the first PNA, the explicit relations for the gauge functions up to the second PNA are derived. The harmonic gauge functions at this order do not satisfy the above relation, as is also obvious from I. However, a 'quasi-harmonic' gauge condition can be chosen, with only small changes in one kind of term in the harmonic gauge functions, satisfying the required relations. In the quasi-harmonic gauge, the equations of motion have the same functional form as in the harmonic gauge.

In order to extend those results to higher orders of approximation, i.e. $2\frac{1}{2}$-, 3-PNA etc, the quasi-harmonic gauge could be used. However, as it is generally accepted, the reaction effect due to the gravitational wave radiation must appear at the order c^{-5} ($2\frac{1}{2}$-PNA) of the equations of motion. To take account of the gravitational radiation effect, the post-Newtonian scheme of Chandrasekhar must be altered by including a radiation condition in the far zone. It must be pointed out that the PNA scheme breaks down at the far zone because it is based in the first-order substitution of the Laplacian operator (instantaneous potential) for the d'Alembert operator (retarded or advanced potential), which is only valid in the near zone. A different approximation to the field equations must be used in the far zone and the solutions in this zone must be finally matched with the solutions in the near zone. For a discussion of these problems, see Burke and Thorne (1970), Burke (1971) and Ehlers *et al* (1976). Chandrasekhar and Esposito (1970) evaluated the gravitational radiation effect, using the Sommerfeld radiation condition and imposing the harmonic gauge condition on the field equations.

2. Restrictions to the gauge functions imposed by Poincaré invariance

The Einstein field equations,

$$R_{\mu\nu} = -(8\pi G/c^4)(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}) \qquad (\mu, \nu = 0, 1, 2\ \ 3) \tag{2.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ the energy–momentum tensor, $g_{\mu\nu}$ the metric tensor and G the gravitational constant, can be solved in a slow-motion approximation scheme. This method is based on the expansion of the metric tensor in powers of c^{-1}:

$$g_{\mu\nu} \simeq \sum_{L=0}^{N} c^{-L} \overset{L}{g}_{\mu\nu} \equiv \overset{(N)}{g}_{\mu\nu} \qquad \left(\overset{0}{g}_{\mu\nu} \equiv \eta_{\mu\nu}\right) \tag{2.2}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowskian metric tensor. The notation $\overset{L}{f}$ and $\overset{(L)}{f}$ in the sense of (2.2) will have the following meaning in the rest of the paper: $\overset{L}{f}$ without brackets is one of the coefficients of the c^{-1} expansion and $\overset{(L)}{f}$ is the approximation to the actual function f.

The basic assumption (expansion in powers of c^{-1}) involved in the slow approximation implies that the PNA is only valid in the 'near zone' whose retarded potentials can be approached by instantaneous potentials, the retarded term appearing always in the next step of approximation.

The field equations (2.1) can be written, up to order N, in the form of Poisson-like equations:

$$\Delta \overset{(N)}{g}_{ij} = \overset{(N)}{W}_{i,j} + \overset{(N)}{W}_{j,i} + \overset{(N)}{S}_{ij} \qquad (i, j = 1, 2, 3) \tag{2.3a}$$

$$\Delta \overset{(N+1)}{g}_{0i} = \overset{(N+1)}{W}_{,i} + \overset{(N)}{W}_{i,0} + \overset{(N+1)}{S}_{0i} \tag{2.3b}$$

$$\Delta \overset{(N+2)}{g}_{00} = 2\overset{(N+1)}{W}_{,0} - \overset{(N)2}{g_{ij}\, g_{00,ij}} - \overset{(N)2}{W_i g_{00,i}} + \overset{(N+2)}{S}_{00} \tag{2.3c}$$

where $\overset{N}{W}_i \equiv \overset{N}{g}_{ik,k} - \tfrac{1}{2}\overset{N}{g}_{kk,i}$, $\overset{N+1}{W} \equiv \overset{N+1}{g}_{0k,k} - \tfrac{1}{2}\overset{N}{g}_{kk,t}$, the subindex $_{,i}$ stands for $\partial/\partial x^i$ and $_{,0}$ for $c^{-1}\partial/\partial t$, $\Delta \equiv \partial^2/\partial x^i\, \partial x^i$ is the Laplace operator and the functions $\overset{(N)}{S}_{ij}$, $\overset{(N+1)}{S}_{0i}$ and $\overset{(N+2)}{S}_{00}$ depend on $\overset{(L)}{g}_{ij}$, $\overset{(L+1)}{g}_{0i}$, $\overset{(L+2)}{g}_{00}$ and $\overset{(N-2)}{T}_{\mu\nu}$, $L < N$ (we assume that the energy–momentum tensor begins at order c^2). The integrability conditions of (2.3a) and (2.3b) lead to the conclusion that the functions $\overset{N}{W}_i$ and $\overset{N+1}{W}$ can be considered arbitrary functions (see I); we shall call them *gauge functions*.

For given $\overset{N}{W}_i$ and $\overset{N+1}{W}$ we can obtain, by an iterative method from equations (2.3), the instantaneous potentials $\overset{N}{g}_{ij}$, $\overset{N+1}{g}_{0i}$ and $\overset{N+2}{g}_{00}$. Such potentials are obviously found, with the arbitrariness of harmonic functions which must be determined as a consequence of matching the near zone with the outer zone.

Our purpose is to require the metric tensor to be invariant, up to order N, under a Poincaré transformation. The invariance of the metric tensor under a three-dimensional rotation is easily obtained by requiring $\overset{(N)}{S}_{ij}$, $\overset{(N+1)}{S}_{0i}$ and $\overset{(N+2)}{S}_{00}$ to be tensor, vector and scalar functions respectively at each order. Note that $\overset{N}{W}_i$ and $\overset{N+1}{W}$ must be a vector and a scalar function respectively. The invariance under space and time translations is also easily obtained since the potentials $\overset{(N)}{g}_{ij}$, $\overset{(N+1)}{g}_{0i}$ and $\overset{(N+2)}{g}_{00}$ can be derived, from the Poisson-like equations (2.3), in a form dependent only on relative positions between field and source and without explicit time dependence.

The invariance under a pure Lorentz transformation of velocity $\boldsymbol{V}$ must be considered in detail: let two coordinate systems, labelled by x^μ: $(\boldsymbol{x}, ct)$ and $\xi^\mu : (\boldsymbol{\xi}, c\tau)$, be connected by a pure Lorentz transformation up to order N. This transformation can be expressed by

$$x^i \simeq \sum_{L=0}^{N} c^{-L} \overset{Li}{\psi} \equiv \overset{(N)i}{\psi} \qquad \left(\overset{0i}{\psi} = \xi^i - V^i \tau\right)$$
$$t \simeq \sum_{L=0}^{N+2} c^{-L} \overset{L}{\eta} \equiv \overset{(N+2)}{\eta} \qquad (\overset{0}{\eta} = \tau). \tag{2.4a}$$

At zero order we have the pure Galilean transformation of velocity $\boldsymbol{V}$. The functions $\overset{Li}{\psi}$ and $\overset{L}{\eta}$ $(L \geqslant 2)$ are given by the expressions

$$\overset{Li}{\psi} \equiv \overset{L}{\gamma}(\boldsymbol{\xi} \cdot \boldsymbol{V}/V^2 - \tau) V^i \qquad L \geqslant 2$$
$$\overset{L}{\eta} \equiv \overset{L}{\gamma}\tau - \overset{L-2}{\gamma}(\boldsymbol{\xi} \cdot \boldsymbol{V}) \qquad L \geqslant 2 \tag{2.4b}$$

where $\overset{L}{\gamma}$ comes from

$$(1 - v^2/c^2)^{-1/2} \simeq \sum_{L=0}^{N} c^{-L} \overset{L}{\gamma}. \tag{2.4c}$$

The parameter $\boldsymbol{V}$ has, up to order N, the physical meaning of a relative uniform velocity between the two frames:

$$\boldsymbol{\xi} = \boldsymbol{V}\tau + \mathrm{O}(c^{-(N+2)}) \qquad \boldsymbol{x} = -\boldsymbol{V}t + \mathrm{O}(c^{-(N+2)}).$$

Now, we introduce the following notation: a function $f(x^\mu)$ (generally a potential) will be written f and the same functional form in terms of the new coordinates, $f(\xi^\mu)$, will be written f'. Moreover, when the function f must be written in terms of the new coordinates, using the transformation law (2.4), we will write

$$f \simeq \sum_{L=0}^{N} c^{-L} \overset{\nu}{\underset{L}{f}} \equiv \overset{\nu}{\underset{(N)}{f}} \qquad \overset{\nu}{\underset{0}{f}} \equiv f' + f''. \tag{2.5}$$

The first expression defines $\overset{\nu}{\underset{L}{f}}$ (function of the new coordinates ξ^μ), and the second one defines f''. This last function depends only on the pure Galilean transformation and is not affected by the non-preservation of simultaneity attached to a Lorentz transformation.

The condition of metric tensor invariance under a pure Lorentz transformation $L^\alpha{}_\beta$ up to order N can be expressed in the form

$$\overset{(N)}{g'_{ij}} = \overline{\overset{(N)}{L^\alpha{}_i L^\beta{}_j g_{\alpha\beta}}} + O(c^{-(N+1)}) \qquad \overset{(N+1)}{g'_{0i}} = \overline{\overset{(N+1)}{L^\alpha{}_0 L^\beta{}_i g_{\alpha\beta}}} + O(c^{-(N+2)}) \tag{2.6}$$

$$\overset{(N+2)}{g'_{00}} = \overline{\overset{(N+2)}{L^\alpha{}_0 L^\beta{}_0 g_{\alpha\beta}}} + O(c^{-(N+3)}).$$

The role played by the gauge functions in the transformation law is better emphasised using, instead of (2.6), their Laplacian form

$$\Delta'(\overset{(N)}{g'_{ij}} - \overline{\overset{(N)}{L^\alpha{}_i L^\beta{}_j g_{\alpha\beta}}}) = O(c^{-(N+1)}) \tag{2.7a}$$

$$\Delta'(\overset{(N+1)}{g'_{0i}} - \overline{\overset{(N+1)}{L^\alpha{}_0\, L^\beta{}_i\, g_{\alpha\beta}}}) = O(c^{-(N+2)}) \tag{2.7b}$$

$$\Delta'(\overset{(N+2)}{g'_{00}} - \overline{\overset{(N+2)}{L^\alpha{}_0\, L^\beta{}_0\, g_{\alpha\beta}}}) = O(c^{-(N+3)}) \tag{2.7c}$$

where $\Delta' = \partial^2/\partial\xi^i\,\partial\xi^i$. As the metric tensor coefficients are arbitrary up to the addition of harmonic functions, equations (2.7) do not give any further restriction on the functions $\overset{N}{W_i}$ and $\overset{N+1}{W}$.

Now, equations (2.7) have to be written in terms of the new coordinates: the Lorentz transformation L^α_β has to be written in terms of its c^{-1} expansion according to (2.4), starting at first order with 1, $c^{-1}V^i$ and δ^i_j for L^0_0, L^i_0 and L^i_j respectively. The notation (2.5) has to be used, and we can see that the higher-order potentials $\overset{N}{g'_{ij}}$, $\overset{N+1}{g'_{0i}}$ and $\overset{N+2}{g'_{00}}$ cancel in the three respective equations (2.7) but not the corresponding $\overset{N}{g''_{ij}}$, $\overset{N+1}{g''_{0i}}$ and $\overset{N+2}{g''_{00}}$ which, after using the field equation (2.5), will include the gauge functions.

It is convenient, for our iterative scheme, to write on the left-hand side the dependence on $\overset{(N)}{W''_{i,j}}$, $\overset{(N+1)}{W''_{,i}}$ and $\overset{(N+1)}{W''_{,0}}$ respectively for the three equations (2.7) and which correspond to the use of the Lorentz transformation at the first order. We leave on the right-hand side most of the rest of the terms and call them $\overset{(N)}{A'_{ij}}$, $\overset{(N+1)}{A'_i}$ and $\overset{(N+2)}{A'}$ Equations (2.7) can then be written in the form

$$\overset{(N)}{W''_{i,j}} + \overset{(N)}{W''_{j,i}} = \overset{(N)}{A'_{ij}} \tag{2.8a}$$

$$[\overset{(N+1)}{W''} - c^{-1}V^j(\overset{(N)}{W'_j} + \overset{(N)}{W''_j})]_{,i} + \overset{(N)}{W''_{i,0}} = \overset{(N+1)}{A'} \tag{2.8b}$$

$$[\overset{(N+1)}{W''} - c^{-1}V^j(\overset{(N)}{W'_j} + \overset{(N)}{W''_j})]_{,0} - \tfrac{1}{2}\overset{(N)}{W''_i}\overset{2}{g'_{00,i}} = \overset{(N+2)}{A'} \tag{2.8c}$$

where the subindex $_{,i}$ under a function of the new coordinates ξ^μ stands for $\partial/\partial\xi^i$, and $_{,0}$ for $c^{-1}\,\partial/\partial\tau$.

Equations (2.8) are a system of recurrence equations which give conditions on the gauge functions $\overset{N}{W}'_i$ and $\overset{N+1}{W}'$ (or equivalently $\overset{N}{W}_i$ and $\overset{N+1}{W}$). Equations (2.8*a*) are a differential system for the function $\overset{N}{W}''_i$. From it, assuming integrability, we completely obtain $\overset{N}{W}''_i$ (with the addition of arbitrary harmonic functions which are not considered). The solution of the homogeneous equation has the form $a^i(\tau)+e^{ijk}\xi^j b^k(\tau)$, which is harmonic, and its contribution to $\overset{N}{W}''_i$ is not considered.

Once the function $\overset{N}{W}''_i$ is known, the function $\overset{N}{W}'_i$ is partially known: according to the pure Galilean transformation we only know the terms depending on the velocities of the sources. We call ${}_{\mathrm{s}}\overset{N}{W}'_i$ the 'static' part of $\overset{N}{W}'_i$ for which ${}_{\mathrm{s}}\overset{N}{W}''_i=0$.

Then, from (2.8*b*) and (2.8*c*) we can write

$$[\overset{N+1}{W}''-V^j{}_{\mathrm{s}}\overset{N}{W}'_j]_{,i}=\overset{N+1}{f'_i}\qquad [\overset{N+1}{W}''-V^j{}_{\mathrm{s}}\overset{N}{W}'_j]_{,\tau}=\overset{N+2}{f'}\tag{2.9}$$

where $\overset{N}{f}_i$ and $\overset{N}{f}$ are known. The integrability conditions of this system are $\mathrm{curl}(\overset{N+1}{f'})=0$ and $\overset{N+1}{f'_{i,t}}=\overset{N+2}{f'_{,i}}$. In this case a function $\overset{N+1}{g'}$ is completely determined and is related to the gauge functions by

$$\overset{N+1}{g'}=\overset{N+1}{W}''-V^i{}_{\mathrm{s}}\overset{N}{W}'_i.\tag{2.10}$$

Only gauge functions satisfying (2.8*a*) and (2.10) give a metric tensor invariant under a Poincaré transformation up to order *N*.

This iterative method can be followed in the successive post-Newtonian approximations and we will explore it up to the second PNA for the specific case of a bounded perfect fluid.

3. Restrictions at the first post-Newtonian approximation

In this section we explicitly find the conditions that the gauge functions must satisfy in order to have a metric tensor Poincaré invariant in the first PNA for a perfect fluid.

A perfect fluid is defined in general relativity by the energy–momentum tensor

$$T_{\mu\nu}=\rho(c^2+\Pi+p/\rho)u_\mu u_\nu-pg_{\mu\nu}$$

where p is the pressure, u_μ the four-velocity of an element of the fluid, $\rho\Pi$ the internal energy and $\rho(c^2+\Pi)$ the total mass density as measured in the rest frame of the matter and which is separated into the non-varying part of the rest mass plus the relativistic mass associated with the internal energy of the fluid.

At the zero order $N=0$ (Newtonian approximation) the metric is given, considering that $\overset{-2}{T}_{00}=\rho c^2$, by

$$\overset{(2)}{g}_{00}=1-2U/c^2\qquad \overset{(1)}{g}_{0i}=0\qquad \overset{(0)}{g}_{ij}=-\delta_{ij}\tag{3.1a}$$

where U is the Newtonian potential, the solution of the field equation at order $N=0$:

$$\Delta U=-4\pi G\rho.\tag{3.1b}$$

At this order no election of gauge is necessary and the metric tensor is invariant under the approximate Poincaré transformation, whose Lorentzian part is

$$\overset{(0)}{\psi} = \xi^i - V^i\tau \qquad \overset{(2)}{\eta} = \tau + c^{-2}(V^2/2\tau - \boldsymbol{\xi} \cdot \boldsymbol{V}). \tag{3.2}$$

In this case equations (2.7) imply $U'' = 0$, which is automatically satisfied since ρ is a scalar, $\rho(x^\mu) = \rho'(\xi^\mu)$:

$$\Delta U = \Delta' U' \tag{3.3}$$

and $\Delta = \Delta' + O(c^{-2})$.

At order $N = 1$, the field equations are homogeneous and we choose the usual gauge in which $\overset{1}{g}_{ij} = \overset{2}{g}_{0i} = \overset{3}{g}_{00} = 0$.

Then, at the first PNA, order $N = 2$, we have

$$\overset{2}{S}_{ij} = \overset{2}{g}_{00,ij} + 8\pi G\rho\delta_{ij} \qquad \overset{3}{S}_{0i} = -16\pi G\rho v^i \tag{3.4}$$

$$\overset{4}{S}_{00} = 16\pi G\rho(v^2 - U + \tfrac{1}{2}\Pi + \tfrac{3}{2}p/\rho) + 2(U_{,i})^2$$

where $v^i = \mathrm{d}x^i/\mathrm{d}t$.

The gauge functions $\overset{2}{W}_i$ and $\overset{3}{W}$ are restricted by equations (2.8*a*) and (2.9), being the right-hand sides of them:

$$\overset{2}{A}{}'_{ij} = 0 \qquad \overset{3}{f}{}'_i = -2V^jU'_{,ji} \qquad \overset{4}{f}{}' = -2V^jU'_{,j\tau}. \tag{3.5}$$

Details of the deduction of those functions are left to the Appendix.

The function $\overset{2}{W}_i$ must satisfy

$$\overset{2}{W}{}''_i = 0 \tag{3.6}$$

which implies that $\overset{2}{W}_i \equiv {}_s\overset{2}{W}_i$ is a static function. The integrability condition of equations (2.9) is obviously satisfied and we obtain $\overset{3}{g}{}' = -2V^jU'_{,j}$, and the final restriction (2.10) is

$$\overset{3}{W}{}'' - V^j\overset{2}{W}{}'_j = -2V^jU'_{,j}. \tag{3.7}$$

Only gauge functions satisfying (3.7) give a Poincaré-invariant metric tensor at the required order $N = 2$.

Clearly the 'standard' gauge (Will 1974) $\overset{2}{W}_i = U_{,i}$, $\overset{3}{W} = 0$ does not satisfy it. We can choose an infinite set of functions satisfying those equations, but there are two which give a diagonal space-metric tensor $\overset{(2)}{g}_{ij}$:

$$\overset{2}{W}_j = U_{,j} \qquad \overset{3}{W} = -U_{,t} \tag{3.8}$$

and

$$\overset{2}{W}_j = 0 \qquad \overset{3}{W} = -2U_{,t}.$$

We can also choose a gauge in which the term $\partial^2/\partial t^2$ in the field equation for $\overset{4}{g}_{00}$ is removed (that is one of the advantages of the standard gauge):

$$\overset{2}{W}_j = 2U_{,j} \qquad \overset{3}{W} = 0.$$

It can easily be shown that the gauge condition (3.8) is the harmonic gauge condition which comes from the de Donder condition

$$(\sqrt{-g}\, g^{\mu\nu})_{,\nu} = 0 \tag{3.9}$$

up to the first PNA. In order to extend the calculations to the next order of approximation, we will choose the harmonic gauge at the first PNA.

4. Restrictions at the second post-Newtonian approximation

In this section we assume the metric tensor has been put into the harmonic gauge up to the first PNA, and we find the restrictions to the gauge functions imposed by Poincaré invariance up to the second PNA ($N=4$).

At order $N=3$ the metric equations are homogeneous and we choose a gauge in which $\overset{3}{g}_{ij} = \overset{4}{g}_{0i} = \overset{5}{g}_{00} = 0$.

So, at order $N=4$ we begin with the metric tensor at $N=2$ in the harmonic gauge

$$\overset{2}{g}_{ij} = -2U\delta_{ij} \qquad \overset{3}{g}_{0i} = 4U_i \qquad \overset{4}{g}_{00} = 2U^2 - 4\Phi + \chi_{,tt} \tag{4.1a}$$

where

$$\Delta U_i = -4\pi G\rho v^i \qquad \Delta\Phi = -4\pi G\rho(v^2 + U + \tfrac{1}{2}\Pi + \tfrac{3}{2}p/\rho) \qquad \Delta\chi = -2U. \tag{4.1b}$$

The functions $\overset{4}{S}_{ij}$, $\overset{5}{S}_{0i}$ and $\overset{6}{S}_{00}$ are given (see I) by

$$\begin{aligned}\overset{4}{S}_{ij} = {} & 16\pi G\rho[v^i v^j - v^2\delta_{ij} - (2p/\rho)\delta_{ij}] - 2\Delta(U^2 + 2\Phi)\delta_{ij} - 4\Phi_{,ij} - 2\delta_{ij}U_{,tt} \\ & - 4(U_{i,j} + U_{j,i})_{,t} + \chi_{,ttij} - 4UU_{,ij}\end{aligned} \tag{4.2a}$$

$$\overset{5}{S}_{0i} = -16\pi G\rho[(v^2 + 4U + \Pi + p/\rho)v^i - 2U_i] - 12U_{,i}U_{,t} + 8U_jU_{,ij} - 8U_{j,i}U_{,j} \tag{4.2b}$$

$$\begin{aligned}\overset{6}{S}_{00} = {} & 16\pi G\rho[v^2(v^2 + 4U + \Pi + p/\rho) - (U^2 + 2\Phi) + \tfrac{1}{2}\chi_{,tt}] - 8(U_{,t})^2 \\ & + 12U_{,i}\Phi_{,i} + 8U_{,i}U_{i,t} - 12U(U_{,i})^2 - 16U_{i,j}(U_{i,j} - U_{j,i}) - 3U_{,i}\chi_{,tti}.\end{aligned} \tag{4.2c}$$

The restrictions imposed by the Poincaré invariance, equations (2.8a) and (2.9), are now given by $\overset{4}{A'}_{ij}$, $\overset{5}{f'}_i$ and $\overset{6}{f'}$ (details of their calculation can be found in the Appendix), which are

$$\begin{aligned}\overset{4}{A'}_{ij} = {} & 4\Phi''_{,ij} + 4V^k(U'_{i,kj} + U'_{j,ki}) - 4V^iU'_{,j\tau} - 4V^jU'_{,i\tau} - 4V^k(V^iU'_{,kj} \\ & + V^jU'_{,ki}) - 2V^k\chi'_{,\tau ij} - V^kV^l\chi_{,klij}\end{aligned} \tag{4.3a}$$

$$\overset{5}{f'}_i = \overset{5}{g'}_{,i} \qquad \overset{6}{f'} = \overset{5}{g'}_{,\tau} \tag{4.3b}$$

where

$$\overset{5}{g'} \equiv -2\Phi''_{,\tau} - V^j(4\Phi'_{,j} + 2\Phi''_{,j} + 4U'_{j,\tau} - \chi'_{,\tau j} - \tfrac{3}{2}V^k\chi'_{,\tau kj} - \tfrac{1}{2}V^kV^l\chi'_{,klj} - 4U'U'_{,j}). \tag{4.3c}$$

At this point it is useful to look at the gauge functions of the harmonic gauge, which we will denote by $\overset{4}{W}_i(H)$ and $\overset{5}{W}(H)$, and which can be obtained by the expansion of the de Donder condition (3.9) up to the second PNA (see I):

$$\overset{4}{W}_i(H) = 2\Phi_{,i} + 4U_{i,t} - \tfrac{1}{2}\chi_{,tti} + 2UU_{,i} \tag{4.4a}$$

$$\overset{5}{W}(H) = -2\Phi_{,t} + \tfrac{1}{2}\chi_{,ttt} - UU_{,t} - 8U_iU_{,i} - \tfrac{1}{2}U_{,i}\chi_{,ti}. \tag{4.4b}$$

We suggest a classification of the terms of these gauge functions according to the power in G they exhibit. From (3.1b) and (4.1b) we see that this power corresponds to the number of the potentials U, χ, Φ and U_i that each term has. We will write $\overset{4}{W}_i(H;1)$ for one potential, $\overset{4}{W}_i(H;2)$ for two potentials and the same for $\overset{5}{W}(H)$. According to the properties of these functions, when they are written in terms of the new coordinates using the Galilean transformations, $\overset{4}{W}_i(H;2)$ is a static term because it does not include velocities or time derivatives; therefore $\overset{4}{W}''_i(H;2) = 0$.

From (4.3a) we see that the gauge function $\overset{4}{W}_i$ required for Poincaré invariance has the one-potential term $\overset{4}{W}_i(1)$:

$$\overset{4}{W}_i(1) = \overset{4}{W}_i(H;1). \tag{4.5a}$$

From this and using the notation $\overset{5}{W} = \overset{5}{W}(1) + \overset{5}{W}(2)$, equation (2.10) with $\overset{5}{g}'$ defined in (4.3c) implies that

$$\overset{5}{W}(1) = \overset{5}{W}(H;1). \tag{4.5b}$$

The final form of equation (2.10) can be expressed as

$$\overset{5}{W}''(2) - V^j\overset{4}{W}'_j(2) = 4V^jU'U'_{,j}. \tag{4.5c}$$

Only gauges satisfying the restrictions (4.5) will give the metric tensor invariant under a Poincaré transformation up to the second PNA. It is remarkable that the one-potential terms of the harmonic gauge functions satisfy the restrictions (4.5) but the two-potential terms do not, as can be easily checked. Moreover, this is enough to maintain the Poincaré invariance of the equations of motion in the harmonic gauge, as was shown in I.

An infinite set of functions $\overset{5}{W}(2)$ and $\overset{4}{W}_i(2)$ satisfying (4.5c) can be chosen. We will define a 'quasi-harmonic' gauge by the condition

$$\overset{4}{W}_i(2) = \overset{4}{W}_i(H;2); \tag{4.6a}$$

thus from (4.5c)

$$\overset{5}{W}(2) = 6UU_{,t}. \tag{4.6b}$$

Using that gauge, the space component of the metric tensor $\overset{4}{g}_{ij}$ has the same functional form as that in the harmonic gauge, and also the equations of motion have in both gauges the same functional form. That relation between the harmonic gauge and the quasi-harmonic gauge at the second PNA is similar to the relation between the standard gauge and the harmonic gauge at the first PNA, which also have the same functional form for both $\overset{2}{g}_{ij}$ and the equations of motion.

Due to the form of the two-potential terms in the harmonic gauge functions, a further calculation at higher order would give non-Poincaré-invariant equations of motion. To obtain that invariance the gauge conditions (4.6) could be used.

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Appendix

The first part of this Appendix is devoted to the deduction of expressions (3.5).

The transformation (2.4b) which we must use now includes terms up to $\overset{2}{\psi}{}^i = \frac{1}{2}V^i(\boldsymbol{\xi}\cdot\boldsymbol{V} - V^2\tau)$ and $\overset{4}{\eta} = \frac{3}{8}V^4\tau - \frac{1}{2}(\boldsymbol{\xi}\cdot\boldsymbol{V})V^2$. From (2.7$a$), the evaluation of $\overset{2}{A}{}'_{ij}$ needs only as much work as was needed in the Newtonian approximation for $U''=0$. From (2.7b) and (2.7c) we need to know the parts of $\Delta'\overset{3}{g}{}''_{0i}$ and $\Delta'\overset{4}{g}{}''_{00}$ which do not include the gauge, that is $\overset{3}{S}{}''_{0i}$ and $\overset{4}{S}{}'_{00}$ (note that $\overset{2}{g}{}''_{ij}=0$). Those functions only include the Galilean transformation, so we only need $\partial/\partial x^i = \partial/\partial\xi^i + V^i\,\partial/\partial\tau + O(c^{-2})$, $\partial/\partial t = \partial/\partial\tau + O(c^{-2})$, $v^i = w^i - V^i + O(c^{-2})$ where $w^i \equiv d\xi^i/d\tau$. Moreover, from equation (2.7c) we also need a term which comes from writing the Newtonian potential $\overset{2}{g}_{00}$ in the new coordinates up to the first PNA, which is $\Delta'\underset{2}{\overset{\overset{2}{\nu}}{g}}_{00}$ (or $\Delta'\underset{2}{\overset{\nu}{U}}$). To evaluate it we use (3.3), taking into account that $U = U' + c^{-2}\underset{2}{\overset{\nu}{U}}$ and the expansion of the Laplacian operator is $\Delta = \Delta' + c^{-2}\underset{2}{\overset{\nu}{\Delta}}$. This leads to

$$\Delta'\underset{2}{\overset{\nu}{U}} = -V^iV^jU'_{,ij} - 2V^jU'_{,j\tau}. \tag{A1}$$

The rest of this Appendix is devoted to sketching some details of the calculations leading to (4.3).

The transformation (2.4b) at $N=4$ includes the terms

$$\overset{4}{\psi}{}^i = -\tfrac{3}{8}V^4V^i\tau + \tfrac{3}{8}V^2(\boldsymbol{\xi}\cdot\boldsymbol{V})V^i \qquad \overset{6}{\eta} = \tfrac{5}{16}V^6\tau - \tfrac{3}{8}(\boldsymbol{\xi}\cdot\boldsymbol{V})V^4.$$

This transformation must be substituted in equations (2.7) and all the functions there must be expressed in terms of the new coordinates. For equation (2.7a) we only need the calculation of $\Delta'\overset{4}{g}{}''_{ij}$, which is easily done from the field equations for $\overset{4}{g}_{ij}$ using the Galilean transformation (note that $\Phi'' = -2V^iU'_i + V^2U'$ and $U''_i = -V^iU'_i$) and $\Delta'\underset{2}{\overset{\overset{2}{\nu}}{g}}_{ij}$, which has been calculated in § 3; the rest of the terms have the same functional form in the new and the old coordinates. Those are the calculations leading to $\overset{4}{A}{}'_{ij}$.

For (2.7*b*) only the $\Delta' \overset{3}{\underset{2}{\overset{\nu}{g}}}_{0i}$ needs to be mentioned. The evaluation of $\Delta' \underset{2}{\overset{\nu}{U}}_i$ is made from the definition of U_i, equation (4.1*b*). Substituting u^i in terms of w^i and V^i up to the post-Newtonian order, and similarly for the operator Δ, we finally obtain

$$\Delta' \underset{2}{\overset{\nu}{U}}{}'_i = -4\pi G\rho'(\xi^\mu)[(\boldsymbol{w}\,.\,\boldsymbol{V})w^i - \tfrac{1}{2}(\boldsymbol{w}\,.\,\boldsymbol{V})V^i - \tfrac{1}{2}V^2 w^i]$$
$$-2V^k U'_{i,\tau k} + V^k V^l V^i U'_{,kl} + 2V^k V^i U'_{,\tau k} - V^k V^l U'_{i,kl}.$$

The rearrangement of this and the rest of the terms leads to the first of equations (4.3*b*).

The more laborious terms in (2.7*c*) are $\Delta' \overset{4}{\underset{4}{\overset{\nu}{g}}}_{00}$, $\dot{\Delta}' \underset{2}{\overset{\nu}{U}}$ and $\Delta' \overset{6}{\underset{4}{g}}{}''_{00}$. For the first one we must use the field equation for $\underset{4}{g}_{00}$ at the first PNA in the harmonic gauge and change the coordinates up to the post-Newtonian order. We also need $\underset{2}{\overset{\nu}{U}}$, which from (A1) and (4.1*b*) can be expressed by

$$\underset{2}{\overset{\nu}{U}} = \tfrac{1}{2}V^i V^j \chi'_{,ij} + V^i \chi'_{,\tau i}.$$

For $\Delta' \underset{4}{\overset{\nu}{U}}$ we must use (3.3), expanding Δ up to the second post-Newtonian order, and $U = U' + c^{-2}\underset{2}{\overset{\nu}{U}} + c^{-4}\underset{4}{\overset{\nu}{U}}$; this yields

$$\Delta' \underset{4}{\overset{\nu}{U}} = -V^2 V^i V^j U'_{,ij} - 2V^2 V^i U'_{,\tau i} - V^2 U'_{,\tau\tau} - \tfrac{1}{2}V^i V^j V^k V^l \chi'_{,ijkl}$$
$$-2V^i V^j V^k \chi'_{,\tau ijk} - 2V^i V^j \chi'_{,\tau\tau ij}.$$

Finally, for $\Delta' \overset{6}{g}{}''_{00}$: in the field equation for $\overset{6}{g}_{00}$ we have the term $\overset{4}{g}_{ij} U_{,ij}$ and consequently we need to know $\overset{4}{g}{}''_{ij}$. This term can easily be obtained by integrating the field equation for $\overset{4}{g}_{ij}$ only for the non-static terms, all of which are linear in the potentials Φ, U, U_i, χ. Then the second of equations (4.3*b*) is obtained after a long but straightforward calculation.

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